A new proof of the global Fuchsian-Cauchy problem in Gevrey spaces

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1. Introduction – We consider the Fuchsian Cauchy problem associated to linear partial differential equations with Fuchsian principal part of order m and weight μ in the sense of M. S. Baouendi and C. Goulaouic [1]. We obtain existence and uniqueness of a global solution to this problem in the space of holomorphic functions with respect to the fuchsian variable t and in Gevrey spaces with respect to the other variable x. The method of proof is based on the application of the fixed point theorem in some Banach spaces defined by majorant functions that are suitables to this kind of equations. We introduce new majorant functions as in [3]and [4] which allow us to simplify the proof given in [2] by introducing a new parameter ρ. Our study is limited to Gevrey class defined by H.Komatsu [5]. This same technique has enabled us in [4] to give global resolution for some nonlinear equations of Fuchs type in these same Gevrey classes.

2. Results and Discussion - We study Fuchsian linear partial differential equations in the space $\mathbb{C} \times \mathbb{R}^n$. We denote by t the generic point of $\mathbb{C}$ and by $x = (x_1, \dots, x_n)$ the generic point of $\mathbb{R}^n$. Let Ω be an open set in $\mathbb{R}^n$. For a multiindice $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ we denote $D^\alpha = D_1^{\alpha_1} \dots D_n^{\alpha_n}$ where $D_j = D_{x_j}$ is the partial derivative with respect to x_j and by $|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_n$.
Let $m \geq 1$ be an integer, we denote by E a subset of $\{(l, \alpha) \epsilon \mathbb{N} \times \mathbb{N}^n;\ l + |\alpha| \leq m,\ \ l < m,\ \ \alpha \neq 0\}$.
Let $0 \leq \mu \leq m$, we consider the Cauchy problem

$$\begin{cases} a(t, D_t)u(t,x) = \sum_{(l,\alpha)\epsilon E} a_{(l,\alpha)}(t,x)\, t^{v+1+l-\mu} D_t^l D^\alpha u(t,x) \ + f(t,x); & (t,x) \in \mathbb{C} \times \Omega, \\ D_t^j u(t,x) = w_j(x), \qquad 0 \leq j < \mu, \quad x \in \Omega, \end{cases} \tag{1}$$

where $a(t, D_t)$ is the linear differential operator defined by $a(t, D_t) = \sum_{l=\mu}^m a_l t^{l-\mu} D_t^l$, and a_l for $\mu \leq l \leq m$ are complex constants with $a_m \neq 0$. $a(t, D_t)$ is then a Fuchsian principal part of order m and weight μ. $v = v(l)$ is the integer number defined by $v = \max(\ \mu - l - 1, 0)$ and the coefficients $a_{(l,\alpha)}$ for $(l,\alpha)\epsilon E$ are polynomial functions with respect to x of order strictly inferior to $|\alpha|$ with holomorphic cofficients in $\mathbb{C}_t$.
$or\ (l,\alpha)\epsilon E,\ \ a_{(l,\alpha)}(t,x) =$
$\sum_{|\beta|<|\alpha|} a_{l\alpha\beta}(t)\, x^\beta$ *where* $a_{l\alpha\beta}$ *is an holomorphic function in* $\mathbb{C}_t$.   (2)

We associate to the operator $a(t, D_t)$ the polynomial $P(\lambda) = \sum_{l=\mu}^m a_l \prod_{j=0}^{l-1}(\lambda - j)$ and we consider $\prod_\phi = 1$.We obtain: $t^\mu\, a(t, D_t) = \sum_{l=\mu}^m a_l t^l D_t^l = P(tD_t)$. Then $P(tD_t)$ is a Fuchsian principal part of weight 0.

3. Conclusions - The coefficients $a_{(l,\alpha)}$ assumed verifying (2), then we obtain the following theorem:
THEOREM 1: If $P(\lambda) \neq 0$ for every integer $\lambda \geq \mu$, then for any functions $w_j \in G^{(d)}(\Omega)$, $(0 \leq j \leq \mu)$ and $f \in G^{(\omega,d)}(\mathbb{C} \times \Omega)$; the Cauchy problem (1) admits a unique solution $u \in G^{(\omega,d)}(\mathbb{C} \times \Omega)$.

4. References
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